A CONVEXITY-TYPE INVARIANT FOR THE CRITICAL COAGULATION–FRAGMENTATION HAMILTON–JACOBI EQUATION

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ABSTRACT. We study the critical coagulation–fragmentation equation with multiplicative coagulation kernel $a(s, \hat{s}) = s\hat{s}$ and constant fragmentation kernel $b(s, \hat{s}) = 1$. Under the Bernstein transform, mass-conserving solutions correspond to solutions of a singular Hamilton–Jacobi equation studied by Tran and Van (*Comm. Pure Appl. Math.* **75** (2022), no. 6, 1292–1331). Through this correspondence they proved that mass-conserving solutions are unique on the full critical range $0 < m \leq 1$, but could establish their existence only for $0 < m < \frac{1}{2}$. We identify a one-sided, convexity-type invariant that holds for Bernstein-transform data and is propagated by their viscous scheme as a genuine maximum-principle bound. We call it the *half-slope invariant*. It sharpens the curvature barrier and thereby extends mass-conserving existence to the entire critical range $0 < m \leq 1$. Hence $m = 1$ is the critical mass, confirming the threshold predicted by Vigil and Ziff (*J. Colloid Interface Sci.* **133** (1989), no. 1, 257–264). The same invariant appears in the radial partial-mass formulation of the two-dimensional Keller–Segel equation, whose critical mass is 8π.

1. INTRODUCTION

The coagulation–fragmentation equation describes the evolution of the size distribution of a population of clusters subject to binary coalescence and binary break-up. The coagulation mechanism dates back to Smoluchowski [Smo16]. The mathematical theory of the full equation, covering existence, uniqueness, and the long-time behaviour of solutions, has since been developed extensively. We refer the reader to Ball–Carr [BC90] and the monograph of Banasiak–Lamb–Laurençot [BLL19] for a comprehensive account.

A defining feature of these equations is that solutions need not conserve mass. When coagulation dominates, mass can be lost to clusters of infinite size in finite time, a phenomenon known as *gelation*, while strong fragmentation can produce clusters of size zero [EMP02; Esc+03]. For the borderline multiplicative coagulation kernel $a(s, \hat{s}) = s\hat{s}$ and constant fragmentation kernel $b \equiv 1$, Escobedo–Laurençot–Mischler–Perthame [Esc+03] singled out a critical regime in which the occurrence of mass conservation or gelation is governed by the *size of the initial mass* rather than by the kernels. A balance computation of Vigil–Ziff [VZ89] for the zeroth moment predicts the critical mass to be $m = 1$: globally mass-conserving solutions are expected for initial mass $m \leq 1$, and gelation for $m > 1$. Proving this dichotomy, and in particular the existence of mass-conserving solutions throughout the critical range $0 < m \leq 1$, has remained open. Existence has been available only for small mass: Laurençot [Lau20] obtained mass-conserving solutions for $0 < m < \frac{1}{4 \log 2}$, subsequently improved to $0 < m < \frac{1}{2}$ in [TV22]. For arbitrary $m > 0$

Date: July 2, 2026.

with finite second moment, Tran and the author constructed in [TV23] a unique local-in-time mass-conserving solution, so on the critical range the question is whether the local solution extends globally.

On other fronts, the large time behaviour of solutions of (1.6) was studied by Mitake, Tran, and the author [MTV21], with complete characterizations of the stationary solutions and optimal conditions for convergence.

A fruitful approach to this borderline problem goes back to Menon and Pego [MP04], who introduced the *Bernstein transform*, a desingularized Laplace transform, to analyse Smoluchowski's coagulation equation. The idea is to convert the nonlocal coagulation–fragmentation equation into a *local* equation for the transform. Under mass conservation the Bernstein transform $F(x,t) = \int_0^\infty (1-e^{-sx})c(s,t)\,\mathrm{d}s$ satisfies a singular Hamilton–Jacobi equation, (1.6) below. In [TV22], Tran and the author developed a theory of viscosity solutions for this equation, which lies outside the classical Crandall–Lions framework because of the singular zeroth-order term F/x. They used it to prove uniqueness of mass-conserving solutions on the entire critical range $0 < m \leq 1$, existence on the subrange $0 < m < \frac{1}{2}$, and non-existence for $m > 1$. The existence proof rests on a one-sided curvature bound $xF_{xx} \geq -1$ for the viscosity solution, and it is precisely the derivation of this bound that has been confined to $m < \frac{1}{2}$.

The purpose of this work is to remove that confinement. We identify a single a priori estimate, namely a one-sided, convexity-type invariant $W := 2M - xM_x \geq 0$ for the function $M := mx - F$. This estimate holds automatically for Bernstein-transform initial data and is propagated by the viscous approximations of [TV22]. It sharpens the bound on the coefficient governing the curvature barrier, replacing $B \leq 2M_x$ by $B \leq M_x$. The factor of two thereby saved is exactly what the restriction $m < \frac{1}{2}$ was paying for. The curvature barrier $xF_{xx} > -1$ therefore holds on the entire range $0 < m \leq 1$, and we finally confirm the predicted critical mass $m = 1$.

We recall from [TV22] the notion of solution we work with.

Definition 1.1 (Weak solution in the measure sense)**.** For each $t \geq 0$, let $c_t(\,\mathrm{d}s)$ be a positive Radon measure on $(0,\infty)$. We say that $(c_t)_{t\geq 0}$ is a *weak solution in the measure sense* of (1.1) with the kernels (1.4) if, for every test function $\phi \in BC([0,\infty)) \cap \mathrm{Lip}([0,\infty))$ with $\phi(0) = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \phi(s)\,c_t(\,\mathrm{d}s) = \frac{1}{2}\int_0^\infty\int_0^\infty \big(\phi(s+\hat{s}) - \phi(s) - \phi(\hat{s})\big)\,s\hat{s}\,c_t(\,\mathrm{d}s)\,c_t(\,\mathrm{d}\hat{s}) - \frac{1}{2}\int_0^\infty\int_0^s \big(\phi(s) - \phi(\hat{s}) - \phi(s-\hat{s})\big)\,\mathrm{d}\hat{s}\,c_t(\,\mathrm{d}s).$$

The solution is *mass-conserving* if $m_1(t) = \int_0^\infty s\,c_t(\,\mathrm{d}s) = m_1(0)$ for all $t \geq 0$.

Theorem 1.2. *Let $c_0 = c(\cdot,0)$ be a nonnegative measure on $(0,\infty)$ with $m_1(0) = m \in (0,1]$ and bounded zeroth and second moments, that is,*

$$m_0(0) = \int_0^\infty c_0(s)\,\mathrm{d}s < \infty \qquad \text{and} \qquad m_2(0) = \int_0^\infty s^2 c_0(s)\,\mathrm{d}s < \infty.$$

Then the coagulation–fragmentation equation (1.1) has a unique mass-conserving weak solution in the measure sense of Definition 1.1. For $m > 1$, by contrast, (1.1) has no global mass-conserving solution. Hence $m = 1$ is the critical mass.

1.1. **The model and its Bernstein transform.** The continuous coagulation–fragmentation equation for the cluster-size density $c(s,t) \geq 0$, $s,t \geq 0$, is

$$\partial_t c(s,t) = Q_c(c)(s,t) + Q_f(c)(s,t), \tag{1.1}$$

where, for nonnegative symmetric coagulation and fragmentation kernels a and b, the coagulation and fragmentation operators are

$$Q_c(c)(s,t) = \frac{1}{2}\int_0^s a(y,s-y)\,c(y,t)\,c(s-y,t)\;\mathrm{d}y - c(s,t)\int_0^\infty a(s,y)\,c(y,t)\;\mathrm{d}y, \tag{1.2}$$

$$Q_f(c)(s,t) = -\frac{1}{2}\,c(s,t)\int_0^s b(s-y,y)\;\mathrm{d}y + \int_0^\infty b(s,y)\,c(y+s,t)\;\mathrm{d}y. \tag{1.3}$$

Throughout we take the multiplicative coagulation kernel and constant fragmentation kernel

$$a(s,\hat{s}) = s\hat{s}, \qquad b(s,\hat{s}) = 1. \tag{1.4}$$

Following [TV22], we apply the *Bernstein transform*

$$F(x,t) \overset{\text{def}}{=} \int_0^\infty \left(1 - e^{-sx}\right) c(s,t)\,\mathrm{d}s, \qquad (x,t) \in [0,\infty)^2. \tag{1.5}$$

Writing $m_1(t) = \int_0^\infty s\,c(s,t)\,\mathrm{d}s$ for the total mass and assuming mass conservation $m_1(t) \equiv m$, the transform F solves the singular Hamilton–Jacobi equation

$$\begin{gathered} F_t + \tfrac{1}{2}\,(F_x - m)(F_x - m - 1) + \frac{F}{x} - m = 0 \quad \text{in } (0,\infty)^2, \\ 0 \leq F \leq mx, \qquad F(x,0) = F_0(x). \end{gathered} \tag{1.6}$$

The constraint $0 \leq F \leq mx$ is intrinsic to the transform: $c \geq 0$ gives $F \geq 0$, while $1 - e^{-sx} \leq sx$ gives $F \leq mx$. The Hamiltonian

$$H(p,z,x) = \tfrac{1}{2}(p-m)(p-m-1) + \frac{z}{x} - m$$

is monotone but not Lipschitz in z as $\partial_z H = 1/x \to +\infty$ as $x \to 0^+$, so (1.6) lies outside the classical Crandall–Lions theory. In [TV22], the authors extended the classical theory of viscosity solutions to study (1.6).

Vigil and Ziff [VZ89] predicted that the critical mass should be $m = 1$. Their computation is based on the following simple calculation. Integrating (1.1) against $\phi \equiv 1$ gives, for the zeroth moment $m_0(t) = \int_0^\infty c(s,t)\,\mathrm{d}s$,

$$\frac{\mathrm{d}}{\mathrm{d}t} m_0(t) = \tfrac{1}{2}\,m_1(t)\big(1 - m_1(t)\big). \tag{1.7}$$

If $m_1 \equiv m > 1$, then m_0 becomes negative in finite time, signalling gelation. For $0 < m \leq 1$ the zeroth moment stays nonnegative.

1.2. **The known gap.** We recall the relevant results of [TV22] (theorem numbers refer to that paper). On the full critical range there is uniqueness:

> *If $0 < m \leq 1$ and F_0 is Lipschitz, sublinear, $0 \leq F_0 \leq mx$, then (1.6) has a unique Lipschitz sublinear viscosity solution, whence (1.1) has at most one mass-conserving solution.*

For $m > 1$ there is non-existence: (1.6) admits no sublinear C^1 solution, so (1.1) has no global mass-conserving solution. The existence side, however, is established only on a strict subrange:

> *If F_0 is the Bernstein transform of c_0 with $m_1(0) = m \in (0, \frac{1}{2})$ and bounded zeroth and second moments, then (1.1) has a mass-conserving weak (measure) solution.*

Thus the window $\frac{1}{2} \le m \le 1$ is open. The purpose of this work is to isolate a single a priori estimate which, once established, removes the artificial factor $\frac{1}{2}$ and brings existence up to the critical mass.

On the discrete side, Jang and Tran [JT25] state the critical-mass conjecture in the form used here and prove existence of mass-conserving solutions exactly on the range $0 < m < \frac{1}{2}$, so the discrete analogue of the window $\frac{1}{2} \le m \le 1$ remains open as of this writing.

1.3. **Standing hypotheses on the initial datum.** Throughout we adopt conditions in [TV22] on the Bernstein-transform datum F_0, recorded here so that the present note is self-contained.

Assumption 1.3 (Conditions on the initial datum, after [TV22])**.** There exist $\beta \in (0, 1)$ and $C > 0$ such that

- (A1) $0 \le F_0'(x) \le m$ and $F_0'(0) = m$,
- (A2) $-C \le F_0''(x) \le 0$,
- (A3) $-\dfrac{m}{e} \le xF_0''(x) \le 0$ and $\left\|xF_0''\right\|_{C^{0,\beta}([0,\infty))} \le C$.

These conditions follow whenever F_0 is the Bernstein transform of a nonnegative measure c_0 with mass $m_1(0) = m$ and bounded second moment $m_2(0) = \int_0^\infty s^2\, c_0(s)\, \mathrm{d}s < \infty$. Indeed, differentiating $F_0(x) = \int_0^\infty (1 - e^{-sx}) c_0(s)\, \mathrm{d}s$ gives $xF_0''(x) = -\int_0^\infty s\, \varphi(sx)\, c_0(s)\, \mathrm{d}s$ with $\varphi(z) = ze^{-z} \in [0, e^{-1}]$, whence $-m/e \le xF_0'' \le 0$. Moreover φ is 1-Lipschitz and bounded, so $|\varphi(sx) - \varphi(sy)| \le (2/e)^{1-\beta} \big(s\,|x - y|\big)^\beta$ for any $\beta \in (0, 1)$. Hence

$$\left|xF_0''(x) - yF_0''(y)\right| \le (2/e)^{1-\beta}\, |x - y|^\beta \int_0^\infty s^{1+\beta} c_0(s)\, \mathrm{d}s,$$

and $\int_0^\infty s^{1+\beta} c_0\, \mathrm{d}s \le m^{1-\beta}\, m_2(0)^\beta < \infty$ by interpolation, which bounds $\|xF_0''\|_{C^{0,\beta}}$ and gives (A3). In Theorem 1.2 we additionally assume $m_0(0) < \infty$, which gives $F_0 \in L^\infty$ and is used in the localization at spatial infinity (Proposition 3.6).

1.4. **Main idea.** Introduce

$$M(x, t) = mx - F(x, t), \qquad q(x, t) = M_x(x, t) = m - F_x(x, t). \tag{1.8}$$

A direct substitution (Section 2) turns (1.6) into the clean form

$$M_t = \tfrac{1}{2}\, q(q + 1) - \frac{M}{x}\,, \qquad q = M_x. \tag{1.9}$$

The estimate we propagate is the convexity-type, one-sided bound

$$W := 2M - xM_x \ge 0 \qquad \Longleftrightarrow \qquad \frac{M}{x} \ge \frac{1}{2}\, M_x\,. \tag{1.10}$$

A function for which the property $W \geq 0$ holds for all time is said to possess the *half-slope invariant*. This is exactly the inequality that the curvature-barrier argument of [TV22] did not exploit. We show three things. First, (1.10) holds at $t = 0$ for any Bernstein-transform datum (Lemma 3.1). Second, it is propagated by the inviscid flow (Lemma 3.2) and, rigorously, by the viscous δ-regularization of [TV22] (Proposition 3.6 and Corollary 3.8). Third, it upgrades the bound on the dangerous coefficient in the barrier from $B \leq 2M_x$ to $B \leq M_x$, which is what the $\frac{1}{2}$-threshold was paying for (Section 4).

1.5. **A curious Keller–Segel analogue.** The half-slope invariant has a curious analogue in the radial Keller–Segel equation, where the same one-sided quantity satisfies a linear parabolic equation with no zeroth-order term, see (6.6). This is purely an aside, recorded in Section 6. It plays no role in the proof.

1.6. **Organization.** Section 2 carries out the substitution $M = mx - F$ and records the resulting equation for M. Section 3 discusses the half-slope invariant of M: it is verified at $t = 0$ for Bernstein-transform data and then shown to propagate, first under the inviscid flow, and then, as a genuine maximum-principle estimate, under the viscous δ-regularization of [TV22]. Section 4 uses the invariant to sharpen the bound on the dangerous coefficient to $B \leq M_x$, thereby extending the curvature barrier $xF_{xx} > -1$ to the entire range $0 < m \leq 1$. Section 5 proves Theorem 1.2. Finally, Section 6 records the Keller–Segel analogue of the half-slope invariant.

2. The transformed equation

Start from (1.6) and set $M = mx - F$, so that

$$F = mx - M, \qquad F_x = m - M_x, \qquad F_t = -M_t.$$

Then $F_x - m = -M_x$ and $F_x - m - 1 = -M_x - 1$, while $\frac{F}{x} - m = \frac{mx-M}{x} - m = -\frac{M}{x}$. Substituting,

$$-M_t + \tfrac{1}{2}(-M_x)(-M_x - 1) - \frac{M}{x} = 0,$$

which is (1.9). With $q = M_x$ this reads $M_t = \frac{1}{2}q(q+1) - M/x$.

For Bernstein data the new unknown has a transparent representation. If $F_0(x) = \int_0^\infty (1 - e^{-sx}) c_0(s)\,\mathrm{d}s$ and $m = \int_0^\infty s\, c_0(s)\,\mathrm{d}s$, then

$$M_0(x) = mx - F_0(x) = \int_0^\infty \big(sx - 1 + e^{-sx}\big) c_0(s)\,\mathrm{d}s, \tag{2.1}$$

$$M_{0,x}(x) = m - F_{0,x}(x) = \int_0^\infty s\big(1 - e^{-sx}\big) c_0(s)\,\mathrm{d}s. \tag{2.2}$$

Because $sx \geq 1 - e^{-sx} \geq 0$ and $1 - e^{-sx} \geq 0$, it follows that $M_0 \geq 0$ and $M_{0,x} \geq 0$. The latter is the statement $0 \leq F_{0,x} \leq m$. These representations are the basis for the initial invariant.

3. The half-slope invariant

3.1. **Invariant propagation for inviscid flow.** We first note the following fact at $t = 0$.

Lemma 3.1. *Let F_0 be the Bernstein transform of a nonnegative measure c_0 on $(0,\infty)$ with finite first moment $m = \int_0^\infty s\, c_0(s)\,\mathrm{d}s < \infty$. Then $M_0 = mx - F_0$ satisfies*

$$2M_0(x) - x\, M_{0,x}(x) \ge 0 \qquad \text{for all } x \ge 0.$$

Proof. From (2.1)–(2.2),

$$2M_0 - xM_{0,x} = \int_0^\infty \Big[2(sx-1+e^{-sx}) - x\, s(1-e^{-sx}) \Big] c_0(s)\,\mathrm{d}s = \int_0^\infty \Big[sx-2+(sx+2)e^{-sx} \Big] c_0(s)\,\mathrm{d}s.$$

Set $z = sx \ge 0$ and $\psi(z) = z - 2 + (z+2)e^{-z}$. Then $\psi(0) = 0$ and

$$\psi'(z) = 1 - (z+1)e^{-z} \ge 0 \qquad (z \ge 0),$$

because $z + 1 \le e^z$ for all $z \ge 0$. Hence $\psi \ge 0$ on $[0,\infty)$, and the integrand $\psi(sx)c_0(s)$ is nonnegative. Therefore $2M_0 - xM_{0,x} \ge 0$. □

Throughout, we denote

$$W = 2M - xM_x \qquad \text{and} \qquad q = M_x\,. \tag{3.1}$$

We then have the following identities

$$W_x = M_x - xM_{xx} = q - xq_x\,, \qquad W_{xx} = -x\, M_{xxx} = -x\, q_{xx}\,. \tag{3.2}$$

Lemma 3.2. *Let M be a smooth solution of (1.9) on a region of $(0,\infty)^2$. Then*

$$W_t - \Big(M_x + \tfrac{1}{2}\Big) W_x + \frac{3}{2x}\, W = 0\,. \tag{3.3}$$

Consequently, as long as the solution stays smooth and its characteristics remain in $(0,\infty)$, $W(\cdot,0) \ge 0$ implies $W(\cdot,t) \ge 0$.

Proof. Differentiating (1.9) in x gives $q_t = (q + \frac{1}{2})q_x - q/x + M/x^2$. Therefore,

$$W_t = 2M_t - xq_t = q^2 + 2q - \frac{3M}{x} - x\Big(q + \tfrac{1}{2}\Big)q_x.$$

Since $\big(q + \frac{1}{2}\big)W_x = \big(q + \frac{1}{2}\big)q - x\big(q + \frac{1}{2}\big)q_x$, we get

$$W_t - (q + \tfrac{1}{2})W_x = \tfrac{3}{2}q - \tfrac{3M}{x} = -\tfrac{3}{2x}W\,,$$

which is (3.3). Along a characteristic $\dot{X} = -(q + \frac{1}{2})$ one has

$$\dot{W} = -\tfrac{3}{2X}W\,.$$

As a consequence, W retains its sign along every characteristic that remains in $(0,\infty)$, where the coefficient $3/(2X)$ stays finite. □

Remark 3.3. When $q \ge 0$ the characteristics move to the left with speed at least $\frac{1}{2}$, so each of them reaches $x = 0$ in finite time, and $3/(2X)$ blows up there. Sign preservation survives, because the backward-in-time characteristic through any point of $(0,\infty)^2$ moves away from $x = 0$ and stays in $(0,\infty)$ up to $t = 0$.

3.2. **Viscous approximations.** In [TV22], the authors construct the regular solution as the vanishing-viscosity limit of

$$F_t + \tfrac{1}{2}(F_x - m)(F_x - m - 1) + \frac{F}{x} - m = A_{\varepsilon,\delta}(x)\, F_{xx}, \qquad A_{\varepsilon,\delta}(x) = \varepsilon\, a(x) + \delta,$$
$$F(x,0) = F_0(x), \qquad F(0,t) = 0, \tag{3.4}$$

where $\varepsilon > 0$, $\delta \geq 0$, and $a \in C^\infty([0,\infty))$ is nonnegative, nondecreasing and concave with

$$a(x) = x \ \text{ on } [0,1], \qquad a(x) = 2 \ \text{ on } [3,\infty)\,. \tag{3.5}$$

For clarity, we will always write $A(x)$ for $A_{\varepsilon,\delta}(x)$. In the M-variable, (3.4) becomes

$$M_t = A(x)\, M_{xx} + \tfrac{1}{2}\, M_x(M_x + 1) - \frac{M}{x}. \tag{3.6}$$

Lemma 3.4. *Let M solve (3.6) classically. Then, with the notations (3.1),*

$$W_t - \Big(q + \tfrac{1}{2}\Big) W_x - A\, W_{xx} + \frac{S}{x}\, W_x + \frac{3}{2x}\, W = \frac{S}{x}\, q\,, \tag{3.7}$$

where $S(x) := 2A(x) - xA'(x)$.

Proof. Differentiating (3.6) in x, $q_t = A'q_x + Aq_{xx} + (q + \frac{1}{2})q_x - q/x + M/x^2$. Using $W_t = 2M_t - xq_t$, $W_x = q - xq_x$ and $W_{xx} = -xq_{xx}$, a direct computation gives

$$W_t = q^2 + 2q - \frac{3M}{x} + (2A - xA')q_x - xAq_{xx} - x\Big(q + \tfrac{1}{2}\Big)q_x,$$

and assembling the stated combination, all $q_x, q_{xx}, q^2, M/x$ terms cancel, leaving (3.7). □

Lemma 3.5. *Let a be as in (3.5). One has $2a(x) - xa'(x) \geq 0$ for all $x \geq 0$. Consequently, for $A = \varepsilon a + \delta$,*

$$S(x) = 2A - xA' = \varepsilon\,(2a - xa') + 2\delta \ \geq \ 0.$$

Proof. Let $g(x) = 2a(x) - xa'(x)$. Then $g(0) = 2a(0) = 0$ (as $a(x) = x$ near 0) and $g'(x) = a'(x) - x\, a''(x) \geq 0$, because $a' \geq 0$ (a nondecreasing), $a'' \leq 0$ (a concave) and $x \geq 0$. Thus g is nondecreasing with $g(0) = 0$, so $g \geq 0$. Adding the δ-layer only increases S by $2\delta \geq 0$. □

3.3. **Invariant propagation for the viscous approximations.** We now prove $W \geq 0$ as a genuine maximum-principle estimate, first for the uniformly parabolic problem ($\delta > 0$).

Proposition 3.6 (Propagation for $F^{\varepsilon,\delta}$)**.** *Let $0 < m \leq 1$, $\varepsilon > 0$, $\delta > 0$, $A = \varepsilon a + \delta$ with a as in (3.5), and let F_0 be bounded and satisfy (A1)–(A2) with $F_0(0) = 0$. Let $F^{\varepsilon,\delta}$ be the classical solution of (3.4) with $F^{\varepsilon,\delta}(0,t) = 0$, $F^{\varepsilon,\delta}(\cdot,0) = F_0$. Put $M = mx - F^{\varepsilon,\delta}$, $q = M_x$, $W = 2M - xM_x$. If $W(\cdot,0) \geq 0$, then*

$$W(x,t) \geq 0 \quad \textit{for all } x > 0,\ t \geq 0\,.$$

The proof of Proposition 3.6 rests on a gradient bound for $F^{\varepsilon,\delta}$. In [TV22] this bound is their Lemma 3.5. The statement there concerns the degenerate approximation obtained after $\delta \downarrow 0$, and the uniformly parabolic case $\delta > 0$ that we use is treated inside its proof. For self-consistency we give a complete proof, following the scheme of Lemma 3.1 of [TV22] with the localization made explicit.

Lemma 3.7 (Gradient bound for the uniformly parabolic problem)**.** *Let $0 < m \le 1$, $\varepsilon > 0$, $\delta > 0$, and let F_0 satisfy (A1) and (A2) with $F_0(0) = 0$. Let $F^{\varepsilon,\delta}$ be the classical solution of (3.4). Then $0 \le F^{\varepsilon,\delta} \le mx$ and*

$$0 \le F^{\varepsilon,\delta}_x \le m\,. \tag{3.8}$$

Proof. Write $F = F^{\varepsilon,\delta}$ and $p = F^{\varepsilon,\delta}_x$, and fix a strip $[0,\infty)\times[0,T]$. By (A1) and $F_0(0) = 0$, integration gives $0 \le F_0(x) \le mx$. From the theory of parabolic equations [LSU68], F is smooth on $(0,\infty)^2$, and both F and p are continuous on the strip with $\Lambda_T := \sup_{[0,\infty)\times[0,T]} |p| < \infty$.

Step 1. We show $0 \le F \le mx$. Write

$$N[\phi] := \phi_t + \tfrac{1}{2}(\phi_x - m)(\phi_x - m - 1) + \frac{\phi}{x} - m - A\,\phi_{xx},$$

so that $N[F] = 0$ and $N[mx] = 0$, while $m \le 1$ gives $N[0] = \frac{1}{2}m(m-1) \le 0$. Let w denote either $0 - F$ or $F - mx$. In both cases $w = u - v$ with $N[u] \le N[v]$, and the mean value theorem applied to the quadratic term yields

$$w_t + b\,w_x + \frac{w}{x} - A\,w_{xx} \le 0, \qquad b(x,t) = \xi(x,t) - m - \tfrac{1}{2},$$

where $\xi(x,t)$ lies between u_x and v_x. Since u_x and v_x take the values 0, p or m, we get $|b| \le \Lambda_T + 2m + 1 =: \bar b_0$. Moreover, $w \le 0$ on $\{t = 0\}$, $w = 0$ on $\{x = 0\}$, and $|w| \le (\Lambda_T + m)\,x$.

We claim that $w \le 0$. Set $h(x,t) := e^{\lambda t}(1 + x^2)$ with $\lambda := \bar b_0 + 2(2\varepsilon + \delta)$. Using $2x \le 1 + x^2$, $A \le 2\varepsilon + \delta$ and $e^{\lambda t} \le h$,

$$h_t + b\,h_x + \frac{h}{x} - A\,h_{xx} \;\ge\; \lambda h - \bar b_0(1+x^2)e^{\lambda t} - 2(2\varepsilon+\delta)e^{\lambda t} \;\ge\; 0.$$

Fix $\gamma > 0$ and set $U := w - \gamma h$, so that $U_t + b\,U_x + U/x - A\,U_{xx} \le 0$. As $|w| \le (\Lambda_T + m)x$, $U \to -\infty$ when $x \to \infty$ uniformly in t, so U attains its maximum over the strip at some (x_0, t_0). If this maximum were positive, then $x_0 > 0$ and $t_0 > 0$, because $U \le w \le 0$ on $\{t = 0\}$ and $U = -\gamma e^{\lambda t} < 0$ on $\{x = 0\}$. At such a maximum $U_t \ge 0$, $U_x = 0$, $U_{xx} \le 0$ and $U/x_0 > 0$, whence $U_t + b\,U_x + U/x_0 - A\,U_{xx} > 0$, a contradiction. Hence $U \le 0$, and $\gamma \downarrow 0$ gives $w \le 0$, which proves $0 \le F \le mx$.

Step 2. We identify boundary values of p. By Step 1, $0 \le F(x,t)/x \le m$, so $F(0,t) = 0$ and continuity of p up to $x = 0$ give $p(0,t) = \lim_{x\to 0^+} F(x,t)/x \in [0,m]$. Together with (A1) at $t = 0$, $0 \le p \le m$ on the parabolic boundary $\{t = 0\} \cup \{x = 0\}$ of the strip.

Step 3. Introduce the linear parabolic operator

$$\mathcal{L}^{\varepsilon,\delta}\phi := \phi_t + \Big(p - m - \tfrac{1}{2} - A'\Big)\phi_x - A\,\phi_{xx},$$

Differentiating (3.4) in x and writing, by the mean value theorem and $F(0,t) = 0$, $F(x,t) = x\,p(\theta x, t)$ with $\theta = \theta(x,t) \in (0,1)$, we get

$$\mathcal{L}^{\varepsilon,\delta}p + \frac{p(x,t) - p(\theta x, t)}{x} = 0 \qquad \text{for } x > 0,\ 0 < t \le T.$$

Since $0 \le a' \le 1$ by (3.5) and concavity, the drift coefficient is bounded, with $|p - m - \frac{1}{2} - A'| \le \Lambda_T + m + 1 + \varepsilon =: \bar b$.

Step 4. We show $p \le m$. Fix $\sigma > 0$ and $0 < \nu < \sigma/\bar b$, and consider $\varphi := p - \nu x - \sigma t$ on the strip. As $\varphi \to -\infty$ when $x \to \infty$, the maximum of φ is attained at some (x_0, t_0).

If $x_0 > 0$ and $t_0 > 0$, then $p_t \geq \sigma$, $p_x = \nu$, $p_{xx} \leq 0$, and $\varphi(x_0,t_0) \geq \varphi(\theta x_0,t_0)$ gives $p(x_0,t_0) - p(\theta x_0,t_0) \geq 0$. The first three facts and the drift bound give $\mathcal{L}^{\varepsilon,\delta} p\,(x_0,t_0) \geq \sigma - \bar{b}\,\nu$, the fourth makes the nonlocal term nonnegative, and the equation of Step 3 then forces

$$0 \;=\; \mathcal{L}^{\varepsilon,\delta} p\,(x_0,t_0) + \frac{p(x_0,t_0) - p(\theta x_0,t_0)}{x_0} \;\geq\; \sigma - \bar{b}\,\nu \;>\; 0,$$

a contradiction. Hence the maximum is attained on $\{t=0\} \cup \{x=0\}$, where $\varphi \leq p \leq m$ by Step 2. Therefore $p \leq m + \nu x + \sigma t$, and letting $\nu \downarrow 0$ and then $\sigma \downarrow 0$ gives $p \leq m$.

Step 5. We show $p \geq 0$. Symmetrically, the minimum of $\psi := p + \nu x + \sigma t$ over the strip is attained. At a minimum point with $x_0 > 0$ and $t_0 > 0$, the reversed inequalities $p_t \leq -\sigma$, $p_x = -\nu$, $p_{xx} \geq 0$ and $p(x_0,t_0) - p(\theta x_0,t_0) \leq 0$ turn the equation of Step 3 into $0 \leq -\sigma + \bar{b}\,\nu < 0$, a contradiction. Hence the minimum is attained on $\{t=0\} \cup \{x=0\}$, where $\psi \geq p \geq 0$ by Step 2, and letting $\nu \downarrow 0$ and then $\sigma \downarrow 0$ gives $p \geq 0$. □

Proof of Proposition 3.6. Write the linear operator on the left of (3.7) as

$$\mathcal{L}\phi := \phi_t - \Big(q + \tfrac{1}{2}\Big)\phi_x + \frac{S}{x}\phi_x - A\,\phi_{xx} + \frac{3}{2x}\phi, \qquad \text{so that} \qquad \mathcal{L}W = \frac{S}{x}\,q.$$

By Lemma 3.5, $S \geq 0$, and by (3.8), $q = m - F_x^{\varepsilon,\delta} \geq 0$. Therefore, $\mathcal{L}W = \frac{S}{x}q \geq 0$. The zeroth-order coefficient $3/(2x)$ of $\mathcal{L}$ is nonnegative and the second-order coefficient $-A \leq 0$, so $\mathcal{L}$ is a proper uniformly parabolic operator on $[\eta, R]$ for the minimum principle.

Fix $T > 0$.

Step 1. We first claim that

$$0 \;\leq\; F^{\varepsilon,\delta}(x,t) \;\leq\; K_T := \|F_0\|_{L^\infty} + mT \qquad \text{on } [0,\infty) \times [0,T].$$

The lower bound, together with $F^{\varepsilon,\delta} \leq mx$, is part of Lemma 3.7. For the upper bound, (3.8) gives $F_x^{\varepsilon,\delta} - m \leq 0$ and $F_x^{\varepsilon,\delta} - m - 1 \leq -1 < 0$, so $(F_x^{\varepsilon,\delta} - m)(F_x^{\varepsilon,\delta} - m - 1) \geq 0$. Since also $F^{\varepsilon,\delta}/x \geq 0$ by the lower bound, equation (3.4) yields the differential inequality

$$F_t^{\varepsilon,\delta} \;\leq\; A\,F_{xx}^{\varepsilon,\delta} + m \qquad \text{in } (0,\infty) \times (0,T].$$

Hence $V := F^{\varepsilon,\delta} - \|F_0\|_{L^\infty} - mt$ satisfies

$$V_t \leq A\,V_{xx}, \qquad V(\cdot,0) \leq 0, \qquad V(0,\cdot) \leq 0, \qquad V \leq mx.$$

Here the linear growth $V \leq mx$ comes from $F^{\varepsilon,\delta} \leq mx$, and A is bounded with $0 < \delta \leq A \leq 2\varepsilon + \delta$. We claim that these facts force $V \leq 0$ on $[0,\infty) \times [0,T]$. Fix $\gamma > 0$ and set $U := V - \gamma\big(x^2 + 2(2\varepsilon+\delta)t\big)$. Then $U_t \leq A\,U_{xx}$, because $A \leq 2\varepsilon + \delta$. Moreover, $U \leq 0$ on $\{t=0\}$ and on $\{x=0\}$, and $U \leq mx - \gamma x^2 \leq 0$ wherever $x \geq m/\gamma$. The classical maximum principle on the rectangle $[0, m/\gamma] \times [0,T]$ then forces $U \leq 0$ on all of $[0,\infty) \times [0,T]$. Letting $\gamma \downarrow 0$ gives $V \leq 0$. Thus $0 \leq F^{\varepsilon,\delta} \leq K_T$, as claimed.

Consequently, at the right edge, using $W = 2(mx - F) - x(m - F_x) = mx - 2F + xF_x$ and $F_x \geq 0$,

$$W(x,t) \geq mx - 2K_T, \qquad \text{so} \qquad W(R,t) \geq 0 \;\text{ whenever } R \geq \frac{2K_T}{m}.$$

Step 2. Since $M(0,t) = 0$ and $M_x = q \in [0,m]$,

$$M(x,t) = \int_0^x q(y,t)\,\mathrm{d}y \in [0, mx]\,.$$

Therefore,

$$W(x,t) = 2M - xq \geq -xq \geq -mx.$$

In particular $W(\eta,t) \geq -m\eta$ for every $\eta > 0$.

Step 3 (localized minimum principle). Fix $0 < \eta < R$ with $R \geq 2K_T/m$, and set $Y := W + m\eta$. By Steps 1–2, $Y \geq 0$ on the parabolic boundary $\partial Q_{\eta,R} := \partial((\eta,R)\times(0,T])$. Indeed $Y(x,0) = W(x,0) + m\eta \geq 0$, $Y(\eta,t) \geq -m\eta + m\eta = 0$, and $Y(R,t) \geq 0$. Moreover,

$$\mathcal{L}Y = \mathcal{L}W + \frac{3}{2x}\,m\eta = \frac{S}{x}q + \frac{3}{2x}m\eta \geq 0.$$

If Y had a negative interior minimum at $(x_0,t_0) \in Q_{\eta,R}$, then $Y(x_0,t_0) < 0$, $Y_t(x_0,t_0) \leq 0$, $Y_x(x_0,t_0) = 0$, $Y_{xx}(x_0,t_0) \geq 0$, so

$$\mathcal{L}Y(x_0,t_0) = Y_t - A\,Y_{xx} + \frac{3}{2x_0}Y \leq \frac{3}{2x_0}\,Y(x_0,t_0) < 0,$$

contradicting $\mathcal{L}Y \geq 0$. Hence $Y \geq 0$ on $\overline{Q_{\eta,R}}$, i.e. $W \geq -m\eta$ on $[\eta,R]\times[0,T]$. Letting $\eta \to 0^+$ gives $W \geq 0$ on $(0,R)\times[0,T]$. Since also $W \geq 0$ for $x \geq R$ (Step 1) and R may be taken arbitrarily large, $W \geq 0$ on $(0,\infty)\times[0,T]$. As $T > 0$ was arbitrary, the proof is complete. □

Corollary 3.8 (Propagation for F^ε)**.** *Let $F^\varepsilon = \lim_{\delta\to 0^+} F^{\varepsilon,\delta}$, and $M^\varepsilon = mx - F^\varepsilon$. Then*

$$2M^\varepsilon - xM^\varepsilon_x \geq 0 \quad on\ (0,\infty)\times[0,\infty)\,.$$

Proof. Proposition 3.6 gives $2M^{\varepsilon,\delta} - xM^{\varepsilon,\delta}_x \geq 0$ for every $\delta > 0$. On compact subsets of $(0,\infty)\times(0,\infty)$ the equation is uniformly parabolic as $\delta \to 0$ (because $a(x) > 0$ there), so $F^{\varepsilon,\delta} \to F^\varepsilon$ locally uniformly.

Furthermore, each $F^{\varepsilon,\delta}$ is concave in x ([TV22], proof of Lemma 3.5, where the concavity is established at the level of the δ-problem) and the limit F^ε is C^1 in x ([TV22], Lemma 3.6). By the convex analysis fact that locally uniform convergence of concave functions implies convergence of the derivatives at every point of differentiability of the limit, $F^{\varepsilon,\delta}_x \to F^\varepsilon_x$ pointwise on $(0,\infty)$.

Passing to the limit in $2(mx - F^{\varepsilon,\delta}) - x(m - F^{\varepsilon,\delta}_x)$ yields the claim. □

4. Removal of the $m < \frac{1}{2}$ obstruction

4.1. **The improved barrier.** Throughout this section F^ε denotes the degenerate viscous approximation of [TV22] obtained from (3.4) after $\delta \downarrow 0$, namely the classical solution of

$$F^\varepsilon_t + \tfrac{1}{2}(F^\varepsilon_x - m)(F^\varepsilon_x - m - 1) + \frac{F^\varepsilon}{x} - m = \varepsilon\,a(x)\,F^\varepsilon_{xx}, \qquad F^\varepsilon(x,0) = F_0, \quad F^\varepsilon(0,t) = 0, \tag{4.1}$$

with a the cutoff (3.5). To keep the regularization visible we write

$$G^\varepsilon = xF^\varepsilon_{xx}, \qquad M = mx - F^\varepsilon, \qquad q = M_x = m - F^\varepsilon_x, \qquad W = 2M - xM_x.$$

From Lemma 3.5 and Lemma 3.6 of [TV22],

$$0 \leq F^\varepsilon_x \leq m, \qquad F^\varepsilon \text{ is concave in } x, \qquad G^\varepsilon \leq 0, \qquad G^\varepsilon(0,t) = 0, \tag{4.2}$$

while Corollary 3.8 supplies the propagated invariant

$$W = 2M - xM_x \geq 0, \qquad \text{equivalently} \qquad \frac{M}{x} \geq \frac{M_x}{2}. \tag{4.3}$$

Finally, hypothesis (A3) controls the initial curvature: for Bernstein data $re^{-r} \leq e^{-1}$ gives $xF_0''(x) \geq -m/e$, so for $0 < m \leq 1$

$$G^\varepsilon(\cdot, 0) = xF_0'' \geq -\frac{m}{e} > -1. \tag{4.4}$$

The following theorem replaces Lemma 3.7 of [TV22], extending their curvature barrier from $0 < m < \frac{1}{2}$ to the full critical range $0 < m \leq 1$. The gain comes from the invariant (4.3), which sharpens the zeroth-order coefficient bound from $B \leq 2M_x$ to $B \leq M_x$.

Theorem 4.1 (Improved curvature barrier)**.** *Let $0 < m \leq 1$, and let F^ε be the degenerate viscous approximation* (4.1) *associated with Bernstein-transform data satisfying the hypotheses of Theorem 1.2. If $\varepsilon < \frac{1}{4}$, then*

$$-1 < xF^\varepsilon_{xx} \leq 0 \qquad \text{on } (0,\infty)^2.$$

Proof. We run a localized minimum principle for $G^\varepsilon = xF^\varepsilon_{xx}$, following [TV22], Lemma 3.7 and Remark 3.8, but with the invariant (4.3) in place of the crude bound used there.

Step 1. Put $\alpha(T) = \inf_{(x,t)\in[0,\infty)\times[0,T]} G^\varepsilon(x,t)$. As in [TV22], Lemma 3.7, $G^\varepsilon = xF^\varepsilon_{xx}$ is bounded and Hölder continuous on $[0,\infty) \times [0,T]$ and satisfies $G^\varepsilon \leq 0$, $G^\varepsilon(0,t) = 0$ (cf. (4.2)). Hence α is continuous and nonincreasing in T, and by (4.4), $\alpha(0) \geq -m/e > -1$. It therefore suffices to rule out a first time T with $\alpha(T) = -1$.

We proceed by contradiction and suppose that there is a first time T where $\alpha(T) = -1$.

Step 2. Differentiating (4.1) twice in x,

$$\partial_t F^\varepsilon_{xx} + \Big(F^\varepsilon_x - \big(m + \tfrac{1}{2}\big)\Big)F^\varepsilon_{xxx} + (F^\varepsilon_{xx})^2 + \frac{F^\varepsilon_{xx}}{x} - \frac{2F^\varepsilon_x}{x^2} + \frac{2F^\varepsilon}{x^3} = \varepsilon\big(a''F^\varepsilon_{xx} + 2a'F^\varepsilon_{xxx} + aF^\varepsilon_{xxxx}\big). \tag{4.5}$$

Step 3. Fix a large k. Choose $(y_k, s_k) \in [0,\infty) \times [0,T]$ with

$$G^\varepsilon(y_k, s_k) \leq \alpha(T) + \tfrac{1}{2k},$$

and then $0 < \rho_k < \frac{1}{k}$ so small that $\rho_k y_k \leq \frac{1}{2k}$. Here ρ_k is the localization parameter, distinct from the already-removed regularization δ. Since G^ε is bounded below and $\rho_k x \to +\infty$ as $x \to \infty$, the function $G^\varepsilon(x,t) + \rho_k x$ attains its minimum over $[0,\infty) \times [0,T]$ at some point (x_k, t_k). By minimality and the choice of (y_k, s_k),

$$\alpha(T) \ \leq \ G^\varepsilon(x_k, t_k) + \rho_k x_k \ \leq \ G^\varepsilon(y_k, s_k) + \rho_k y_k \ \leq \ \alpha(T) + \tfrac{1}{k},$$

and since $G^\varepsilon(x_k, t_k) \geq \alpha(T)$ this also gives $\rho_k x_k \leq \frac{1}{k}$.

Furthermore, $G^\varepsilon + \rho_k x$ equals 0 at $x = 0$ and tends to $+\infty$ as $x \to \infty$, while on $\{t = 0\}$ its values are at least $-m/e > \alpha(T) + \frac{1}{k}$ for k large. So, $x_k, t_k > 0$.

Set $\alpha_k = G^\varepsilon(x_k, t_k)$, so $\alpha_k \to -1$. At the localized minimum

$$G^\varepsilon_t \leq 0, \qquad G^\varepsilon_x = -\rho_k, \qquad G^\varepsilon_{xx} \geq 0.$$

We translate these into pointwise relations for the derivatives of F^ε. Since $G^\varepsilon = xF^\varepsilon_{xx}$,

$$G^\varepsilon_x = F^\varepsilon_{xx} + x\,F^\varepsilon_{xxx}, \qquad G^\varepsilon_{xx} = 2F^\varepsilon_{xxx} + x\,F^\varepsilon_{xxxx},$$

and, by the definition $\alpha_k = G^\varepsilon(x_k, t_k) = x_k F^\varepsilon_{xx}$, we have $F^\varepsilon_{xx} = \alpha_k / x_k$ at (x_k, t_k). Evaluating the first-order condition $G^\varepsilon_x = -\rho_k$ there gives $F^\varepsilon_{xx} + x_k F^\varepsilon_{xxx} = -\rho_k$. Multiplying by

x_k and using $x_k F^\varepsilon_{xx} = \alpha_k$,

$$x_k^2 F^\varepsilon_{xxx} = -x_k F^\varepsilon_{xx} - \rho_k x_k = -\alpha_k - \rho_k x_k. \tag{4.6}$$

Similarly the second-order condition $G^\varepsilon_{xx} \geq 0$ reads $2F^\varepsilon_{xxx} + x_k F^\varepsilon_{xxxx} \geq 0$, i.e. $x_k F^\varepsilon_{xxxx} \geq -2F^\varepsilon_{xxx}$. Multiplying by x_k and inserting (4.6),

$$x_k^2 F^\varepsilon_{xxxx} \;\geq\; -2x_k F^\varepsilon_{xxx} = \frac{2\big(\alpha_k + \rho_k x_k\big)}{x_k} = 2F^\varepsilon_{xx} + 2\rho_k. \tag{4.7}$$

Step 4. Multiply (4.5) by x_k^2 and evaluate at (x_k, t_k). On the left-hand side, $x_k^2\,\partial_t F^\varepsilon_{xx} = x_k G^\varepsilon_t \leq 0$ may be dropped, (4.6) turns the third-derivative term into $(m + \frac{1}{2} - F^\varepsilon_x)(\alpha_k + \rho_k x_k)$, and the remaining terms are $x_k^2 (F^\varepsilon_{xx})^2 = \alpha_k^2$, $x_k F^\varepsilon_{xx} = \alpha_k$, and $-2F^\varepsilon_x + 2F^\varepsilon / x_k = B_k$. On the right-hand side, (4.6), (4.7) and $a \geq 0$ give

$$\varepsilon\big(a'' x_k^2 F^\varepsilon_{xx} + 2a' x_k^2 F^\varepsilon_{xxx} + a\, x_k^2 F^\varepsilon_{xxxx}\big) \;\geq\; \varepsilon\,\alpha_k\Big(\frac{2a}{x_k} - 2a' + x_k a''\Big) - 2\varepsilon a' \rho_k x_k + 2\varepsilon a \rho_k,$$

with a, a', a'' evaluated at x_k. Discarding the nonnegative term $2\varepsilon a(x_k)\rho_k$ and moving $-2\varepsilon a' \rho_k x_k$ to the left-hand side, we obtain

$$\alpha_k^2 + \alpha_k\Big(m + \tfrac{3}{2} - F^\varepsilon_x\Big) + B_k + \rho_k x_k\Big(m + \tfrac{1}{2} + 2\varepsilon a'(x_k) - F^\varepsilon_x\Big) \;\geq\; \varepsilon\,\alpha_k\Big(\frac{2a(x_k)}{x_k} - 2a'(x_k) + x_k a''(x_k)\Big), \tag{4.8}$$

where $B_k = 2\big(F^\varepsilon - x_k F^\varepsilon_x\big)/x_k$ and $F^\varepsilon, F^\varepsilon_x, M_x$ are evaluated at (x_k, t_k).

Step 5. The cutoff (3.5) satisfies $a \geq 0$, $a' \geq 0$, $a'' \leq 0$ and $a(x) \leq x$, so $2a/x - 2a' + xa'' \leq 2$. Since $\alpha_k \leq 0$, multiplying this bound by $\varepsilon\alpha_k$ reverses it, so the right-hand side of (4.8) satisfies $\varepsilon\alpha_k\big(2a/x - 2a' + xa''\big) \geq 2\varepsilon\alpha_k$. Absorbing $2\varepsilon\alpha_k$ into the linear coefficient,

$$\alpha_k^2 + \alpha_k\Big(\tfrac{3}{2} + M_x - 2\varepsilon\Big) + B_k + \rho_k x_k\Big(m + \tfrac{1}{2} + 2\varepsilon a'(x_k) - F^\varepsilon_x\Big) \;\geq\; 0. \tag{4.9}$$

Step 6. Since $0 \leq F^\varepsilon_x \leq m \leq 1$, a' is bounded, and $\rho_k x_k \leq 1/k$, the localization term in (4.9) is $o_k(1)$. So,

$$\alpha_k^2 + \alpha_k\Big(\tfrac{3}{2} + M_x - 2\varepsilon\Big) + B_k + o_k(1) \;\geq\; 0. \tag{4.10}$$

Step 7 (the invariant bound). In the M-variable $B_k = 2(F^\varepsilon - x_k F^\varepsilon_x)/x_k = 2\big(M_x - M/x_k\big)$. Concavity of F^ε with $F^\varepsilon(0,t) = 0$ gives $B_k \geq 0$, while the invariant (4.3) gives $M/x_k \geq M_x/2$. Thus,

$$0 \leq B_k \leq M_x\,. \tag{4.11}$$

This single factor of two, giving $B_k \leq M_x$ in place of the crude $B_k \leq 2M_x$ of [TV22], is what the whole extension turns on.

Step 8. By (4.11) and $\alpha_k \leq 0$,

$$\alpha_k^2 + \alpha_k\Big(\tfrac{3}{2} + M_x - 2\varepsilon\Big) + B_k \;\leq\; \alpha_k^2 + \alpha_k\Big(\tfrac{3}{2} - 2\varepsilon\Big) + (1 + \alpha_k) M_x.$$

Since $0 \leq M_x \leq m \leq 1$ and $1 + \alpha_k \to 0$ while $\alpha_k \to -1$,

$$\limsup_{k\to\infty}\Big(\alpha_k^2 + \alpha_k\big(\tfrac{3}{2} + M_x - 2\varepsilon\big) + B_k\Big) \;\leq\; 1 - \Big(\tfrac{3}{2} - 2\varepsilon\Big) = -\tfrac{1}{2} + 2\varepsilon \;<\; 0 \qquad (\varepsilon < \tfrac{1}{4}), \tag{4.12}$$

which contradicts (4.10). Hence $\alpha(T)$ never reaches -1. Together with (4.2), this gives $-1 < xF^\varepsilon_{xx} \leq 0$ on $(0,\infty)^2$. □

Letting $\varepsilon \downarrow 0$, the vanishing-viscosity limit $F = \lim_{\varepsilon\to 0} F^\varepsilon$ inherits

$$-1 \le xF_{xx} \le 0$$

in the same sense, and by the same compactness and stability argument, as in the proof of Theorem 1.7 of [TV22]. Together with $0 \le F_x \le m$, this furnishes the hypotheses of Theorem 5.1.

Remark 4.2 (Discriminant bookkeeping)**.** The argument uses no discriminant condition. Nonetheless the same factor of two is visible at the level of the roots of $P(\zeta) = \zeta^2 + A\zeta + B$ with $A = \frac{3}{2} + M_x - 2\varepsilon$. The improved bound $B \le M_x$ together with $0 \le M_x \le 1$ gives

$$A^2 - 4B \;\ge\; \Big(\tfrac{3}{2} + M_x - 2\varepsilon\Big)^2 - 4M_x = \Big(M_x - \tfrac{1}{2} - 2\varepsilon\Big)^2 + 2 - 8\varepsilon \;>\; 0 \qquad (\varepsilon < \tfrac{1}{4}),$$

whereas the crude bound $B \le 2M_x$ yields a discriminant whose sign degenerates as $m \uparrow \frac{1}{2}$. This degeneration is the root-level shadow of the same loss.

4.2. **Why $m \le 1$ is essential.** The first-crossing bound (4.12) no longer sees m. One might worry that it "proves too much", since for $m > 1$ there is no C^1 solution (Theorem 1.4 of [TV22]). The resolution is that the hypotheses needed even to *run* the argument fail for $m > 1$. The propagation Proposition 3.6 rests throughout on the gradient bound (3.8): it gives $q = M_x \ge 0$ and, in Step 1, both the sign of the quadratic term and the linear growth $F^{\varepsilon,\delta} \le mx$. In turn, (3.8) is proved in Lemma 3.7 from the two-sided bound $0 \le F^{\varepsilon,\delta} \le mx$, where 0 and mx are respectively a subsolution and an exact solution of (3.4), and 0 is a subsolution precisely because

$$\tfrac{1}{2}m(m-1) \le 0 \quad\Longleftrightarrow\quad m \le 1.$$

For $m > 1$ this breaks down as 0 is no longer a subsolution. So the propagation argument of Proposition 3.6 no longer applies and the bound $B \le M_x$ is unavailable.

5. Proof of Theorem 1.2

We start this section by citing a theorem. Its proof is the existence part of Section 3 of [TV22], which is written there for $0 < m < \frac{1}{2}$ and applies on the full range $0 < m \le 1$ after one constant is adjusted. Remark 5.2 below identifies the adjustment.

Theorem 5.1. *Let $0 < m \le 1$, and let F_0 be the Bernstein transform of a nonnegative measure c_0 on $(0,\infty)$ with*

$$m_1(0) = m, \qquad m_0(0) < \infty, \qquad m_2(0) < \infty.$$

Let F be the sublinear viscosity solution of the Hamilton–Jacobi equation (1.6), obtained as the vanishing-viscosity limit in the scheme of [TV22], and assume that the estimates

$$0 \le F_x \le m, \qquad -1 \le xF_{xx} \le 0 \tag{5.1}$$

hold on $(0,\infty)^2$, in the sense obtained from the vanishing-viscosity approximation. Then

$$F \in C^\infty((0,\infty)^2) \cap C^1([0,\infty)^2), \qquad F_x(0,t) = m,$$

and

$$(-1)^{n+1}\,\partial_x^n F \ge 0 \quad \textit{on } (0,\infty)^2 \quad \textit{for every } n \ge 1.$$

Consequently, for each $t > 0$ the map $x \mapsto F(x,t)$ is a Bernstein function and

$$F(x,t) = \int_0^\infty \left(1 - e^{-sx}\right) c_t(\,\mathrm{d}s)$$

for a nonnegative measure c_t on $(0,\infty)$ with

$$\int_0^\infty s\, c_t(\,\mathrm{d}s) = F_x(0,t) = m.$$

The family $(c_t)_{t\geq 0}$, with initial datum c_0, is a mass-conserving weak solution of the coagulation–fragmentation equation (1.1) in the measure sense of Definition 1.1.

Remark 5.2. The existence argument in Section 3 of [TV22] consists of their Propositions 3.9 and 3.10, Lemmas 3.11 and 3.12, and the proof of their Theorem 1.8. It uses the restriction $m < \frac{1}{2}$ in exactly one place. Along the characteristics $X(t)$ of their Proposition 3.9 one has $\dot{X} = \partial_x F - (m + \frac{1}{2})$ with $0 \leq \partial_x F \leq m$, and their bound (3.19) reads $-1 \leq -(m+\frac{1}{2}) \leq \dot{X} \leq -\frac{1}{2}$. The left inequality is exactly the statement $m \leq \frac{1}{2}$. On the full range $0 < m \leq 1$ it must be replaced by

$$-\tfrac{3}{2} \;\leq\; -\Big(m + \tfrac{1}{2}\Big) \;\leq\; \dot{X} \;\leq\; -\tfrac{1}{2}.$$

The estimates in that part of [TV22] which invoke (3.19) through the upper bound $\dot{X} \leq -\frac{1}{2}$, such as their integral bound (3.27) and the proof of their Lemma 3.11, are unaffected, since that bound does not involve m. The lower bound enters only through boundedness, namely in the finite-time absorption of characteristics at $x = 0$ and in the localization windows in the proof of their Proposition 3.10, where the constant -1 becomes $-\frac{3}{2}$ and the windows widen accordingly. The arguments use only that $\dot{X}$ is bounded and strictly negative, so nothing structural changes. We found no other appeal to $m < \frac{1}{2}$ in that part of [TV22]. The remaining occurrences of $\frac{1}{2}$ there are the coefficient $m + \frac{1}{2}$ and numerical constants. Finally, the proof of their Lemma 3.11 uses precisely the bounds $-1 \leq x\partial_x^2 F \leq 0$ of (5.1), which Theorem 4.1 supplies on the full range after $\varepsilon \downarrow 0$.

Sections 3–4 supply the only new input needed beyond [TV22] so that the curvature barrier

$$0 \leq F_x \leq m, \qquad -1 \leq xF_{xx} \leq 0$$

is now valid on the full critical range $0 < m \leq 1$.

5.1. **Proof of Theorem 1.2.** Existence on $0 < m \leq 1$ follows from the barrier (Theorem 4.1) together with Theorem 5.1, which furnishes, for the Bernstein-transform datum F_0 of Theorem 1.2, a nonnegative measure c_t that is a mass-conserving weak solution of (1.1). Uniqueness on $0 < m \leq 1$ is Corollary 1.3 of [TV22]. The non-existence of mass-conserving solutions when $m > 1$ is Corollary 1.5 of [TV22] and has appeared in [BLL19] via a different method. ∎

6. A Keller–Segel curiosity: the same half-slope invariant

This section is interpretive and is not used in the proof. The estimate $2M - xM_x \geq 0$ has a counterpart in the standard radial partial-mass formulation of the two-dimensional parabolic–elliptic Keller–Segel equation. Consider the following parabolic–elliptic Keller–Segel equation

$$\rho_t = \Delta\rho - \nabla\cdot(\rho\nabla c), \qquad -\Delta c = \rho \quad \text{in } \mathbb{R}^2. \tag{6.1}$$

For a radially symmetric solution, the cumulative mass

$$Q(x,t) = \int_{\{|y|^2\le x\}} \rho(y,t)\,\mathrm{d}y \tag{6.2}$$

in the variable $x = r^2$ satisfies the local equation [Bil+06]

$$Q_t = 4x\,Q_{xx} + \frac{1}{\pi}\,Q\,Q_x, \tag{6.3}$$

the diffusion coefficient $4x$ arising from $\partial_r^2 - \frac{1}{r}\partial_r = 4x\,\partial_x^2$ under $x = r^2$. The total mass is $Q(\infty,t)$ and the critical mass is 8π. Normalizing the mass by 8π and rescaling time, the normalized cumulative mass $u = Q/(8\pi)$ satisfies

$$u_t = xu_{xx} + 2uu_x, \tag{6.4}$$

where now the total mass is $\mu = u(\infty,t)$ and $\mu = 1$ is the critical value.

Let

$$P(x,t) = \int_0^x u(s,t)\,\mathrm{d}s, \qquad P_x = u.$$

Integrating (6.4) from 0 to x (using $u(0,t)=0$) gives

$$P_t = xP_{xx} - P_x + P_x^2. \tag{6.5}$$

The Keller–Segel analogue of $W = 2M - xM_x$ is the half-slope quantity $H := 2P - xP_x$.

For a smooth solution, H satisfies a clean equation. One has $H_x = P_x - xP_{xx}$ and $H_{xx} = -xP_{xxx}$, while differentiating (6.5) gives $(P_t)_x = xP_{xxx} + 2P_xP_{xx}$. Hence

$$H_t = 2P_t - x(P_t)_x = -x^2P_{xxx} + 2P_x^2 - 2P_x - 2xP_xP_{xx} + 2xP_{xx},$$

and the right-hand side is exactly $xH_{xx} + 2(P_x - 1)H_x$, so that

$$H_t = xH_{xx} + 2(P_x - 1)H_x. \tag{6.6}$$

Equation (6.6) is linear in H, with drift $2(P_x-1)$ and nonnegative diffusion coefficient x, and $H(0,t) = 0$ because $P(0,t) = 0$ and $u(0,t) = 0$. One might therefore expect the condition $H(\cdot,0) \ge 0$ to be propagated by the maximum principle. Making this rigorous requires handling the degeneracy of the diffusion at $x = 0$ and imposing decay hypotheses on u at $x = \infty$, which we do not pursue in this aside.

A simple sufficient condition for the initial inequality is concavity of the partial mass: if $u_0(0) = 0$ and u_0 is concave, then $u_0(s) \ge \frac{s}{x}u_0(x)$ for $0 \le s \le x$, so

$$P_0(x) = \int_0^x u_0(s)\,\mathrm{d}s \ge \frac{x}{2}u_0(x) = \frac{x}{2}P_{0,x}(x),$$

that is, $2P_0 - xP_{0,x} \ge 0$. Since u_x is proportional to the radial density, this corresponds to a radially nonincreasing density.

The Legendre transform makes the analogy with the coagulation–fragmentation invariant even more striking. For the computations that follow we assume $P_{xx} = u_x > 0$ so that $y = P_x$ is a genuine change of variables and $G_{yy} = 1/P_{xx}$ makes sense. Likewise, we assume $M_{xx} > 0$ on the coagulation–fragmentation side.

Writing $G(y,t) = xy - P(x,t)$ with $y = P_x$, so $G_y = x$ and $P = xy - G$,

$$2P - xP_x \geq 0 \iff 2(xy - G) - xy \geq 0 \iff \frac{G}{G_y} \leq \frac{y}{2}, \tag{6.7}$$

the same half-slope inequality as the Legendre form of our invariant $2M - xM_x \geq 0$.

The two models do not share the same Legendre equation, however. (6.5) transforms to

$$G_t + \frac{G_y}{G_{yy}} = y(1-y), \tag{6.8}$$

whereas the analogous Legendre transform of $M_t = \frac{1}{2}M_x(M_x+1) - M/x$ gives

$$G_t + \frac{G}{G_y} = \tfrac{1}{2}\, y(1-y). \tag{6.9}$$

What they share is the critical polynomial $y(1-y)$ and (potentially) the propagated half-slope inequality $G/G_y \leq \frac{y}{2}$.

Both equations possess a critical mass separating global, mass-conserving (respectively globally smooth) solutions from gelation (respectively blow-up). The critical mass is $m = 1$ for coagulation–fragmentation and 8π (normalized to $\mu = 1$ above) for Keller–Segel. It would be interesting to investigate whether there is a deeper relationship between the two models.

Acknowledgements

The author thanks Nguyen Van Tien, whose talk at the University of Sciences in Ho Chi Minh City on June 29th, 2026 inspired the idea relating the critical coagulation–fragmentation equation to the Keller–Segel equation.

Use of AI tools. Large language models (LLM) were used to verify computations, cross-check the dependence on [TV22], and review drafts. The review identified gaps in earlier versions of Lemma 3.7 and Proposition 3.6. Remark 5.2 was also suggested by LLM in order to help readability. The author implemented and checked all the details and takes full responsibility for all results and proofs.

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